

A REALIZATION OF CERTAIN MODULES FOR THE $N = 4$ SUPERCONFORMAL ALGEBRA AND THE AFFINE LIE ALGEBRA $A_2^{(1)}$

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ABSTRACT. We shall first present an explicit realization of the simple $N = 4$ superconformal vertex algebra $L_c^{N=4}$ with central charge $c = -9$. This vertex superalgebra is realized inside of the $bc\beta\gamma$ system and contains a subalgebra isomorphic to the simple affine vertex algebra $L_{A_1}(-\frac{3}{2}\Lambda_0)$. Then we construct a functor from the category of $L_c^{N=4}$ -modules with $c = -9$ to the category of modules for the admissible affine vertex algebra $L_{A_2}(-\frac{3}{2}\Lambda_0)$. By using this construction we construct a family of weight and logarithmic modules for $L_c^{N=4}$ and $L_{A_2}(-\frac{3}{2}\Lambda_0)$. We also show that a coset subalgebra of $L_{A_2}(-\frac{3}{2}\Lambda_0)$ is an logarithmic extension of the $W(2, 3)$ -algebra with $c = -10$. We discuss some generalizations of our construction based on the extension of affine vertex algebra $L_{A_1}(k\Lambda_0)$ such that $k + 2 = 1/p$ and p is a positive integer.

1. INTRODUCTION

In this paper we explicitly construct certain simple vertex algebras associated to the $N = 4$ superconformal Lie algebra and the affine Lie algebra $A_2^{(1)}$ and apply this construction in the representation theory of vertex algebras. We demonstrate that these vertex algebras have interesting representation theories which include finitely many irreducible modules in the category \mathcal{O} , infinite series of weight irreducible modules and series of logarithmic representations. We will also show that these vertex algebras are connected with logarithmic conformal field theory obtained using logarithmic extension of affine $A_1^{(1)}$ -vertex algebras and higher rank generalizations of triplet vertex algebras.

The $N = 4$ superconformal algebra appeared in the classification of simple formal distribution Lie superalgebras which admit a central extension containing a Virasoro subalgebra with a non-trivial center (cf. [K], [FK]). It is realized by using quantum reduction of affine Lie superalgebras (cf. [KW2], [KRW], [Ar1]). The free-fields realization of the universal vertex algebra associated to $N = 4$ superconformal algebra appeared in [KW2]. In this paper we shall realize the simple $N = 4$ superconformal vertex algebra $L_c^{N=4}$ with central charge $c = -9$. It appears that for this central charge the simple affine vertex algebra $L_{A_1}(-\frac{3}{2}\Lambda_0)$ is conformally embedded into vertex superalgebra $L_c^{N=4}$. Moreover, the Wakimoto module for $L_{A_1}(-\frac{3}{2}\Lambda_0)$ is realized inside of vertex superalgebra $M \otimes F$, where M is a Weyl vertex algebra and F is a Clifford vertex

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superalgebra. We prove that $L_c^{N=4}$ with $c = -9$ is isomorphic to the maximal sl_2 -integrable submodule of $M \otimes F$. In physics terminology vertex superalgebra $M \otimes F$ is called the $\beta\gamma bc$ system. So $N = 4$ superconformal vertex superalgebra is realized as a subalgebra of the $\beta\gamma bc$ system. We classify irreducible $L_c^{N=4}$ -modules in the category of $\frac{1}{2}\mathbb{Z}_{\geq 0}$ -graded modules. It turns out that our classification is similar to that of [AM1]. Vertex operator superalgebra $L_c^{N=4}$ has finitely many irreducible modules in the category \mathcal{O} (in fact only two) and infinitely many irreducible modules in the category of weight modules. Only the category of $\frac{1}{2}\mathbb{Z}_{\geq 0}$ -modules with finite-dimensional weight spaces is semi-simple. By applying the construction from [AM3], we construct a family of logarithmic modules for $L_c^{N=4}$. Next we show that the simple affine vertex operator algebra $L_{A_2}(-\frac{3}{2}\Lambda_0)$ can be realized as a subalgebra of $L_c^{N=4} \otimes F_{-1}$. This construction is similar to that of [A3] where affine vertex algebra of critical level for \widehat{sl}_2 was realized on the tensor product of a vertex superalgebra \mathcal{V} and F_{-1} .

As in [A3], we construct a family of functors \mathcal{L}_s which map (twisted) $L_c^{N=4}$ -modules to untwisted $L_{A_2}(-\frac{3}{2}\Lambda_0)$ -modules. As a consequence, we construct a family of irreducible weight $L_{A_2}(-\frac{3}{2}\Lambda_0)$ -modules and logarithmic modules.

In [A4], we presented an explicit realization of the affine vertex algebras $L_{A_1}(-\frac{4}{3}\Lambda_0)$ and demonstrated that this vertex operator algebra is related with triplet algebra $W(p)$ with $p = 3$ (cf. [AM2]). A connection between $L_{A_1}(-\frac{1}{2}\Lambda_0)$ and triplet algebra $W(2)$ was studied in [R]. Our present construction is also related with \mathcal{W} -algebras appearing in logarithmic conformal field theory. In particular, we show that the parafermion vertex subalgebra $K(sl_3, -\frac{3}{2})$ of $L_{A_2}(-\frac{3}{2}\Lambda_0)$ is an extension of $(1, 2)$ -model for the $W(2, 3)$ -algebras. Moreover, vacuum subspace of $L_{A_2}(-\frac{3}{2}\Lambda_0)$ contains vertex algebra $W_{A_2}(p)$ with $p = 2$ investigated by A. M. Semikhatov in [S1] (see also [AM4]).

For every $p \geq 3$, we also introduce vertex algebra $\mathcal{V}^{(p)}$ which generalize the $N = 4$ superconformal vertex algebra with $c = -9$ and vertex algebra $\mathcal{R}^{(p)}$ which generalize simple affine vertex algebra $L_{A_2}(-\frac{3}{2}\Lambda_0)$.

In our forthcoming publications we shall study fusion rules for modules constructed in this paper.

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2. PRELIMINARIES

In this section we recall the definition of vertex superalgebras, their twisted modules (cf. [FHL], [FLM], [K], [LL]).

Let $(V = V_{\bar{0}} \oplus V_{\bar{1}}, Y, \mathbf{1}, \omega)$ be a vertex operator superalgebra. We shall always assume that

$$V_{\bar{0}} = \coprod_{n \in \mathbb{Z}_{\geq 0}} V(n), \quad V_{\bar{1}} = \coprod_{n \in \frac{1}{2} + \mathbb{Z}_{\geq 0}} V(n)$$

$$\text{where } V(n) = \{a \in V \mid L(0)a = nv\}.$$

For $a \in V(n)$, we shall write $\text{wt}(a) = n$, or $\deg(a) = n$. As usual, vertex operator associated to $a \in V$ is denoted by $Y(a, x)$, with the mode expansion

$$Y(a, x) = \sum_{n \in \mathbb{Z}} a_n x^{-n-1}.$$

Any element $u \in V_{\bar{0}}$ (resp. $u \in V_{\bar{1}}$) is said to be even (resp. odd). We define $|u| = 0$ if u is even and $|u| = 1$ if u is odd. Elements in $V_{\bar{0}}$ or $V_{\bar{1}}$ are called homogeneous. Whenever $|u|$ is written, it is understood that u is homogeneous.

Let σ be canonical automorphism of V of order two.

Assume now that g is an automorphism of the vertex superalgebra V such that g acts semisimply on V and

$$(2.1) \quad V = \oplus_{\bar{\alpha} \in \Gamma/\mathbb{Z}} V^{\bar{\alpha}}$$

$$(2.2) \quad gv = e^{2\pi i \alpha} v, \quad \text{for } v \in V^{\bar{\alpha}}$$

where Γ is an additive subgroup of \mathbb{R} containing \mathbb{Z} and $\bar{\alpha} = \alpha + \mathbb{Z}$. We will always assume that $0 \leq \alpha < 1$.

Definition 2.1. Let g be an automorphism of V be such that (2.1) and (2.2) hold. A weak g -twisted V -module is a pair (M, Y_M) , where $M = M_{\bar{0}} \oplus M_{\bar{1}}$ is a \mathbb{Z}_2 -graded vector space, and $Y_M(\cdot, z)$ is a linear map

$$Y_M : V \rightarrow \text{End}(M)\{z\}, \quad a \mapsto Y_M(a, z) = \sum_{n \in \Gamma} a_n z^{-n-1},$$

satisfying the following conditions for $a, b \in V$ and $v \in M$:

- (M1) $|a_n v| = |a| + |v|$ for any $a \in V$.
- (M2) $Y_M(\mathbf{1}, z) = I_M$.
- (M3) $Y_M(Da, z) = \frac{d}{dz} Y_M(a, z)$.
- (M4) $a_n v = 0$ for $n \gg 0$.
- (M5) $Y_M(a, z) = \sum_{n \in \bar{\alpha}} a_n z^{-n-1}$ for $a \in V^{\bar{\alpha}}$.
- (M6) The twisted Jacobi identity holds

$$\begin{aligned} & z_0^{-1} \delta \left(\frac{z_1 - z_2}{z_0} \right) Y_M(a, z_1) Y_M(b, z_2) - (-1)^{|a||b|} z_0^{-1} \delta \left(\frac{z_2 - z_1}{-z_0} \right) Y_M(b, z_2) Y_M(a, z_1) \\ &= z_2^{-1} \left(\frac{z_1 - z_0}{z_2} \right)^{-\alpha} \delta \left(\frac{z_1 - z_0}{z_2} \right) Y_M(Y(a, z_0)b, z_2). \end{aligned}$$

for $a \in V^{\bar{\alpha}}$, $b \in V$.

Definition 2.2. *An admissible g -twisted V -module is a weak g -twisted V -module M with a grading of the form*

$$M = M(0) \bigoplus_{\mu > 0} M(\mu)$$

such that $M(0) \neq 0$ and for any homogeneous $a \in V$, $n \in \Gamma$ we have

$$a_n M(\mu) \subset M(\mu + \text{wt}(a) - n - 1).$$

In the case $g = 1$, M is called (untwisted) V -module.

Now we shall recall one important construction of twisted modules. Let now $h \in V$ such that

$$L(n)h = \delta_{n,0}h, \quad h(n)h = \delta_{n,1}\gamma \mathbf{1}$$

for any $n \in \mathbb{Z}_{\geq 0}$, where γ is a fixed complex number. Assume that $h(0)$ acts semisimply on V . For simplicity we assume that $h(0)$ has real eigenvalues. Let Γ be an additive subgroup of \mathbb{R} generated by 1 and the eigenvalues of $h(0)$. Then $g_h = e^{2\pi i h(0)}$ is an automorphism of V and (2.1)-(2.2) hold. g_h has finite order if and only if the action of $h(0)$ on V has only rational eigenvalues.

Remark 2.1. *In general the automorphism g can have infinite order, and group Γ/\mathbb{Z} is infinite. Using slightly different terminology, g -twisted modules for Γ/\mathbb{Z} -graded vertex superalgebras were defined and investigated in [KR].*

Define

$$\Delta(h, z) = z^{h(0)} \exp \left(\sum_{n=1}^{\infty} \frac{h(n)}{-n} (-z)^{-n} \right).$$

Applying the results obtained in [Li2] on V -modules we get the following proposition.

Proposition 2.1. *For any weak V -module $(M, Y_M(\cdot, z))$,*

$$(\widetilde{M}, \widetilde{Y}_{\widetilde{M}}(\cdot, z)) := (M, Y_M(\Delta(h, z)\cdot, z))$$

is a g_h -twisted weak V -module. \widetilde{M} is an irreducible twisted V -module if and only if M is an irreducible V -module.

Let us recall the definition of Zhu's algebra for vertex operator superalgebras.

Let g be an automorphism of the vertex operator superalgebra V such that (2.1)- (2.2) hold. Assume also that for the automorphism $g\sigma$ we have

$$(2.3) \quad V = \bigoplus_{\bar{\alpha} \in \widetilde{\Gamma}/\mathbb{Z}} V^{\bar{\alpha}}$$

$$(2.4) \quad g\sigma v = e^{2\pi i \alpha} v, \quad \text{for } v \in V^{\bar{\alpha}}$$

where $\widetilde{\Gamma}$ is an additive subgroup of \mathbb{R} containing \mathbb{Z} and $\bar{\alpha} = \alpha + \mathbb{Z}$ where $0 \leq \alpha < 1$.

Now we recall definition of twisted Zhu's algebra for vertex operator superalgebras (for various versions of the definition see [Z], [Xu], [DZ], [DLM], [DK], [VEk]). For homogeneous $v \in V^{\bar{\alpha}}$

we set $\delta_{\bar{\alpha}} = 1$ if $\bar{\alpha} = \bar{0}$ and $\delta_{\bar{\alpha}} = 0$ otherwise. We define two bilinear maps $*_g : V \times V \rightarrow V$, $\circ_g : V \times V \rightarrow V$ as follows: for homogeneous $a, b \in V$, let

$$(2.5) \quad a *_g b = \begin{cases} \text{Res}_x Y(a, x) \frac{(1+x)^{\text{wt}(a)}}{x} b & \text{if } a, b \in V^{\bar{0}} \\ 0 & \text{if } a \text{ or } b \in V^{\bar{\alpha}}, \bar{\alpha} \neq \bar{0} \end{cases}$$

$$(2.6) \quad a \circ_g b = \text{Res}_x Y(a, x) \frac{(1+x)^{\text{wt}(a)-1+\delta_{\bar{\alpha}}+\alpha}}{x^{1+\delta_{\bar{\alpha}}}} b \quad \text{if } a \in V^{\bar{\alpha}}$$

Next, we extend $*_g$ and \circ_g to $V \otimes V$ linearly, and denote by $O_g(V) \subset V$ the linear span of elements of the form $a \circ_g b$, and by $A_g(V)$ the quotient space $V/O_g(V)$. The space $A_g(V)$ has a unitary associative algebra structure, with the multiplication induced by $*_g$. Algebra $A_g(V)$ is called the g -twisted Zhu algebra of V . The image of $v \in V$, under the natural map $V \mapsto A_g(V)$ will be denoted by $[v]$.

If $g = Id$ we shall denote $A_g(V)$ by $A(V)$.

The following theorem was proved in [DZ] (see also [Z], [DLM], [VEk], [DK]).

Theorem 2.1. *There is a one-to-one correspondence between admissible irreducible g -twisted V -modules and irreducible $A_g(V)$ -modules.*

3. VERTEX OPERATOR ALGEBRA $L(k\Lambda_0)$

In this section we recall some basic facts about vertex operator algebras associated to affine Lie algebras (cf. [FZ], [Li1], [K]).

Let \mathfrak{g} be a finite-dimensional simple Lie algebra over \mathbb{C} and let (\cdot, \cdot) be a nondegenerate symmetric bilinear form on \mathfrak{g} . Let $\mathfrak{g} = \mathfrak{g}_- + \mathfrak{h} + \mathfrak{g}_+$ be a triangular decomposition for \mathfrak{g} . The affine Lie algebra $\hat{\mathfrak{g}}$ associated with \mathfrak{g} is defined as $\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d$, where c is the canonical central element [K] and the Lie algebra structure is given by

$$\begin{aligned} [x \otimes t^n, y \otimes t^m] &= [x, y] \otimes t^{n+m} + n(x, y) \delta_{n+m, 0} c, \\ [d, x \otimes t^n] &= nx \otimes t^n \end{aligned}$$

for $x, y \in \mathfrak{g}$. We will write $x(n)$ for $x \otimes t^n$.

The Cartan subalgebra $\hat{\mathfrak{h}}$ and subalgebras $\hat{\mathfrak{g}}_+$, $\hat{\mathfrak{g}}_-$ of $\hat{\mathfrak{g}}$ are defined by

$$\hat{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}c \oplus \mathbb{C}d, \quad \hat{\mathfrak{g}}_{\pm} = \mathfrak{g} \otimes t^{\pm 1} \mathbb{C}[t^{\pm 1}].$$

Let $P = \mathfrak{g} \otimes \mathbb{C}[t] \oplus \mathbb{C}c \oplus \mathbb{C}d$ be upper parabolic subalgebra. For every $k \in \mathbb{C}$, let $\mathbb{C}v_k$ be 1-dimensional P -module such that the subalgebra $\mathfrak{g} \otimes \mathbb{C}[t] + \mathbb{C}d$ acts trivially, and the central element c acts as multiplication with $k \in \mathbb{C}$. Define the generalized Verma module $N_{\hat{\mathfrak{g}}}(k\Lambda_0)$ as

$$N_{\hat{\mathfrak{g}}}(k\Lambda_0) = U(\hat{\mathfrak{g}}) \otimes_{U(P)} \mathbb{C}v_k.$$

Then $N_{\hat{\mathfrak{g}}}(k\Lambda_0)$ has a natural structure of a vertex algebra generated by fields

$$x(z) = Y(x(-1)\mathbf{1}, z) = \sum_{n \in \mathbb{Z}} x(n) z^{-n-1} \quad (x \in \mathfrak{g}),$$

where $\mathbf{1} = 1 \otimes v_k$ is the vacuum vector.

Let $N_{\mathfrak{g}}^1(k\Lambda_0)$ be the maximal ideal in the vertex operator algebra $N_{\mathfrak{g}}(k\Lambda_0)$. Then $L_{\mathfrak{g}}(k\Lambda_0) = \frac{N_{\mathfrak{g}}(k\Lambda_0)}{N_{\mathfrak{g}}^1(k\Lambda_0)}$ is a simple vertex operator algebra.

If \mathfrak{g} is a simple Lie algebra of type A_n , we shall denote the above vertex operator algebras by $N_{A_n}(k\Lambda_0)$ and $L_{A_n}(k\Lambda_0)$.

Let $M_{\mathfrak{h}}(k)$ denotes the vertex subalgebra of $L_{\mathfrak{g}}(k\Lambda_0)$ generated by fields $h(z)$, $h \in \mathfrak{h}$. Recall that if $k \neq 0$ then $M_{\mathfrak{h}}(k) \cong M_{\mathfrak{h}}(1)$ (cf. [LL]). We have the following coset vertex algebra

$$K(\mathfrak{g}, k) := \text{Com}(M_{\mathfrak{h}}(k), L(k\Lambda_0)) = \{v \in L(k\Lambda_0) \mid h(n)v = 0, h \in \mathfrak{h}, n \geq 0\},$$

which is called the parafermion vertex algebra.

The structure and representation theory of the parafermion vertex algebras in the case when $k \in \mathbb{Z}_{\geq 0}$ have been developed in series of papers [ALY], [DLWY], [DW]. We shall see that for admissible affine vertex algebras the parafermion vertex algebras are related with vertex algebras appearing in logarithmic conformal field theory.

4. LATTICE CONSTRUCTION OF $A_1^{(1)}$ -MODULES OF LEVEL $-\frac{3}{2}$ AND SCREENING OPERATORS

In this section we shall recall the definition of the Wakimoto modules for affine Lie algebra $A_1^{(1)}$ in the case of level $k = -3/2$. For this level we will have nice realization of modules and screening operators using lattice vertex superalgebras and their twisted modules. Details about Wakimoto modules can be found in [Fr], [FB] and [W].

Let $p \in \mathbb{Z}_{>0}$, $p \geq 2$.

Let $L^{(p)}$ be the following lattice

$$L^{(p)} = \mathbb{Z}\alpha + \mathbb{Z}\beta + \mathbb{Z}\delta$$

with the \mathbb{Q} -valued bilinear form $\langle \cdot, \cdot \rangle$ such that

$$\langle \alpha, \alpha \rangle = 1, \quad \langle \beta, \beta \rangle = -1, \quad \langle \delta, \delta \rangle = \frac{2}{p}$$

and other products of basis vectors are zero.

Let $V_{L^{(p)}}$ be the associated generalized vertex algebra (cf. [DL]). If $p = 2$ then $V_{L^{(2)}}$ is a vertex superalgebra.

Assume now that $k + 2 = \frac{1}{p}$.

Define the following three vectors

$$(4.7) \quad e = e^{\alpha+\beta},$$

$$(4.8) \quad h = -2\beta(-1) + \delta(-1),$$

$$(4.9) \quad f = ((k+1)(\alpha(-1)^2 - \alpha(-2)) - \alpha(-1)\delta(-1) + (k+2)\alpha(-1)\beta(-1))e^{-\alpha-\beta}.$$

Then the components of the fields $Y(x, z) = \sum_{n \in \mathbb{Z}} x(n)z^{-n-1}$, $x \in \{e, f, h\}$ satisfy the commutations relations for the affine Lie algebra \widehat{sl}_2 of level k . Moreover, the subalgebra of $V_{L^{(p)}}$ generated by the set $\{e, f, h\}$ is isomorphic to the simple vertex operator algebra $L_{A_1}(k\Lambda_0)$.

Screening operators are

$$(4.10) \quad Q = \text{Res}_z Y(e^{\alpha+\beta-p\delta}, z), \quad \tilde{Q} = \text{Res}_z Y(e^{-\frac{1}{p}(\alpha+\beta)+\delta}, z).$$

They commute with the $\widehat{sl_2}$ -action.

Let M be a subalgebra of $V_{L(p)}$ generated by

$$a = e^{\alpha+\beta}, \quad a^* = -\alpha(-1)e^{-\alpha-\beta}.$$

Then M is isomorphic to the Weyl vertex algebra (cf. [FMS], [Fr], [A2]).

Let $M_\delta(1)$ be the Heisenberg vertex algebra generated by the field $\delta(z) = \sum_{n \in \mathbb{Z}} \delta(n)z^{-n-1}$. Then

$$F_{p/2} = M_\delta(1) \otimes \mathbb{C}[\mathbb{Z}_2^p \delta] \quad \text{and} \quad M \otimes F_{p/2}$$

are subalgebras of $V_{L(p)}$.

We have the following (generalized) vertex algebra

$$\mathcal{V}^{(p)} = \text{Ker}_{M \otimes F_{p/2}} \tilde{Q}.$$

Moreover $L(k\Lambda_0)$ can be realized as a subalgebra of the $M \otimes M_\delta(1) \subset M \otimes F_{p/2}$ (cf. [Fr], [FB], [W]):

$$(4.11) \quad e(z) = a(z),$$

$$(4.12) \quad h(z) = -2 : a^*(z)a(z) : + \delta(z),$$

$$(4.13) \quad f(z) = - : a^*(z)^2 a(z) : + k \partial_z a^*(z) + a^*(z) \delta(z).$$

Since \tilde{Q} commutes with the action of $\widehat{sl_2}$ we have that

$$L_{A_1}(k\Lambda_0) \subset \mathcal{V}^{(p)}.$$

Moreover, one can show that Q acts as a derivation on $\mathcal{V}^{(p)}$. If p is even, then $\mathcal{V}^{(p)}$ is a vertex superalgebra.

Let now $p = 2$ and $k = -3/2$. Let $L = L^{(2)}$. Then V_L is a vertex operator superalgebra and the above formula give a realization of the simple vertex operator algebra inside of V_L .

Under this realization, the Sugawara vector is mapped to

$$(4.14) \quad \omega = \frac{1}{2}(\alpha(-1)^2 - \alpha(-2) - \beta(-1)^2 + \beta(-2) + \delta(-1)^2 - 2\delta(-2)).$$

The components of the field $Y(\omega, z) = \sum_{n \in \mathbb{Z}} L(n)z^{-n-2}$ satisfy the commutation relations for the Virasoro algebra with central charge $c = -9$.

Screening operators are

$$Q = e_0^{\alpha+\beta-2\delta} = \text{Res}_z Y(e^{\alpha+\beta-2\delta}, z), \quad \tilde{Q} = e_0^{-\frac{\alpha+\beta-2\delta}{2}} = \text{Res}_z Y(e^{-\frac{\alpha+\beta-2\delta}{2}}, z).$$

We also note the following important relation:

$$(4.15) \quad \begin{aligned} e(-1)\omega &= e_{-1}^{\alpha+\beta}\omega = \\ &(-(\alpha(-1) + \beta(-1))\beta(-1) + \beta(-2) + \frac{1}{2}\delta(-1)^2 - \delta(-2)) e^{\alpha+\beta}. \end{aligned}$$

Moreover, $F = F_1$ is a Clifford (fermionic) vertex superalgebra and it is generated by $\Psi = e^\delta$, $\Psi^* = e^{-\delta}$ (cf. [K]).

Then formulas (4.11)-(4.13) give a realization of $L(-\frac{3}{2}\Lambda_0)$ as a vertex subalgebra of the Clifford-Weyl vertex superalgebra $M \otimes F$.

Since $e^{\alpha+\beta-2\delta} = a_{-1}\Psi_{-2}^*\Psi^* \in M \otimes F$ we have that

$$Q = \text{Res}_z : a(z) \partial_z \Psi^*(z) \Psi^*(z) : .$$

Since sl_2 acts on $M \otimes F$ by derivations, we have the subalgebra $(M \otimes F)^{int}$ which is the maximal sl_2 -integrable submodule of $M \otimes F$.

Clearly, $L_{A_1}(-\frac{3}{2}\Lambda_0) \subset (M \otimes F)^{int}$. Since e^δ is $\widehat{sl_2}$ -singular vector and since

$$(4.16) \quad f(0)^2 e^\delta = 0$$

we have that $\Psi = e^\delta \in (M \otimes F)^{int}$.

Remark 4.1. *In the physics literature the tensor product vertex algebra $M \otimes F$ is called the $\beta\gamma bc$ system. So we have realized an extension of $L_{A_1}(-\frac{3}{2}\Lambda_0)$ inside of $\beta\gamma bc$ system. A different kind of realization of $L_{A_1}(-\frac{3}{2}\Lambda_0)$ also appeared in [CL]. The authors realized $L_{A_1}(-\frac{3}{2}\Lambda_0)$ inside of the tensor product of three copies of the $\beta\gamma bc$ system. It was shown in [CL] that the simple affine vertex algebra $L_{A_1}(-\frac{3}{2}\Lambda_0)$ (denoted there by $V_{-3/2}(sl_2)$) is connected with Odake's algebra with central charge $c = 9$. In fact Odake's algebra and $V_{-3/2}(sl_2)$ were used to describe certain coset vertex subalgebras of $(M \otimes F)^{\otimes 3}$. Recall that Odake's vertex algebra is an extension of $N = 2$ superconformal vertex algebra with central charge $c = 9$. In the present paper we shall go in a different direction. We shall study $(M \otimes F)^{int}$ and prove that it is isomorphic to the simple $N = 4$ superconformal vertex algebra with central charge $c = -9$, which is a different extension of the $N = 2$ superconformal vertex algebra.*

Let now $\mu \in \mathbb{C}$ and let $g_\mu := e^{2\pi i \mu \delta(0)}$. Then g_μ -is an automorphism of F . Let F^μ be a g_μ -twisted F -module:

$$(F^\mu, \tilde{Y}_{F^\mu}(\cdot, z)) := (F, Y_F(\Delta(\mu\delta(-1)\mathbf{1})\cdot, z)).$$

Remark 4.2. *Recall that the twisted module structure on F^μ is generated by the twisted fields: $z^\mu \Psi(z), z^{-\mu} \Psi^*(z)$ (see [KR]).*

F^μ is untwisted $M_\delta(1)$ -module and we have the following decomposition

$$F^\mu \cong V_{\mathbb{Z}\delta}.e^{\mu\delta} = \bigoplus_{j \in \mathbb{Z}} M_\delta(1).e^{(j+\mu)\delta}.$$

Identifying $g_\mu = Id \otimes g_\mu$, we can consider g_μ as an automorphism of the vertex operator superalgebra $M \otimes F$. Moreover, if \mathcal{M} is any module for the vertex algebra M , then $\mathcal{M} \otimes F^\mu$ is a g_μ -twisted $M \otimes F$ -module. Since $L_{A_1}(-\frac{3}{2}\Lambda_0) \subset M \otimes M_\delta(1)$ we have that $\mathcal{M} \otimes F^\mu$ is an untwisted $L(-\frac{3}{2}\Lambda_0)$ -module. In particular,

$$M \otimes F^\mu = \bigoplus_{j \in \mathbb{Z}} M \otimes M_\delta(1).e^{(\mu+j)\delta},$$

so $M \otimes F^\mu$ is a direct sum of Wakimoto modules for \widehat{sl}_2 of the form $M \otimes M_\delta(1).e^{(\mu+j)\delta}$.

5. A REALIZATION OF THE SIMPLE $N = 4$ SUPERCONFORMAL VERTEX ALGEBRA WITH CENTRAL CHARGE $c = -9$

In this section we shall present an explicit realization of the simple $N = 4$ superconformal vertex algebra $L_c^{N=4}$ with central charge $c = -9$.

We shall first recall the definition of the universal $N = 4$ superconformal vertex algebra $V_c^{N=4}$ of central charge c associated to the $N = 4$ superconformal algebra. Given a vertex superalgebra V and a vector $a \in V$, we expand the field $Y(a, z) = \sum_{n \in \mathbb{Z}} a_n z^{-n-1}$. Define the λ -bracket

$$[a_\lambda b] = \sum_{j \geq 0} \frac{\lambda^j}{j!} a_j b.$$

We shall now define the "small" $N = 4$ superconformal vertex algebra $V_c^{N=4}$ following [K]. The even part of this vertex algebra is generated by the Virasoro field L and three primary fields of conformal weight 1, J^0 , J^+ and J^- . The odd part is generated by four primary fields of conformal weight $\frac{3}{2}$, G^\pm and \overline{G}^\pm .

The remaining (non-vanishing) λ -brackets are

$$\begin{aligned} [J_\lambda^0, J^\pm] &= \pm 2J^\pm & [J_\lambda^0 J^0] &= \frac{c}{3}\lambda \\ [J_\lambda^+ J^-] &= J^0 + \frac{c}{6}\lambda & [J_\lambda^0 G^\pm] &= \pm G^\pm \\ [J_\lambda^0 \overline{G}^\pm] &= \pm \overline{G}^\pm & [J_\lambda^+ G^-] &= G^+ \\ [J_\lambda^- G^+] &= G^- & [J_\lambda^+ \overline{G}^-] &= -\overline{G}^+ \\ [J_\lambda^- \overline{G}^+] &= -\overline{G}^- & [G_\lambda^\pm \overline{G}^\pm] &= (T + 2\lambda)J^\pm \\ [G_\lambda^\pm \overline{G}^\mp] &= & L \pm \frac{1}{2}T J^0 \pm \lambda J^0 + \frac{c}{6}\lambda^2 \end{aligned}$$

Let $V_c^{N=4}$ be the vertex superalgebra freely generated by the fields \overline{G}^\pm , G^\pm , J^+ , J^0 , J^- and L . Let $L_c^{N=4}$ be its simple quotient.

Irreducible highest weight $V_c^{N=4}$ -modules are defined as usual (cf. [KW2] and [Ar1]).

We shall now show that the simple vertex superalgebra $L_c^{N=4}$ with $c = -9$ can be realized as a subalgebra of the lattice vertex superalgebra V_L from Section 4.

Define

$$(5.17) \quad j^+ = e, \quad j^0 = h, \quad j^- = f,$$

$$(5.18) \quad \tau^+ = e^\delta,$$

$$(5.19) \quad \bar{\tau}^+ = Qe^\delta = (\alpha(-1) + \beta(-1) - 2\delta(-1))e^{\alpha+\beta-\delta},$$

$$(5.20) \quad \tau^- = f(0)e^\delta = -\alpha(-1)e^{-\alpha-\beta+\delta},$$

$$(5.21) \quad \bar{\tau}^- = -f(0)Qe^\delta.$$

$$(5.22) \quad \begin{aligned} \omega &= (e(-1)f(-1) + f(-1)e(-1) + \frac{1}{2}h(-1)^2)\mathbf{1} \\ &= \frac{1}{2}(\alpha(-1)^2 - \alpha(-2) - \beta(-1)^2 + \beta(-2) + \delta(-1)^2 - 2\delta(-2)). \end{aligned}$$

(Note that the Virasoro vector ω is the Sugawara Virasoro vector in $L_{A_1}(-\frac{3}{2}\Lambda_0)$).

We can evaluate the above expressions in vertex superalgebra $M \otimes F$.

$$\begin{aligned} \tau^+ &= \Psi, \quad \tau^- = a_{-1}^* \Psi, \\ \bar{\tau}^+ &= 2a_{-1}D\Psi^* + a_{-2}\Psi^*, \\ \bar{\tau}^- &= D^2\Psi^* + (2a_{-1}^*a_{-1} + \delta(-1))D\Psi^* + a_{-1}^*a_{-2}\Psi^*. \end{aligned}$$

Define also the following fields

$$(5.23) \quad J^+(z) = e(z), \quad J^0(z) = h(z), \quad J^- = f(z),$$

$$(5.24) \quad G^\pm(z) = Y(\tau^\pm, z) = \sum_{n \in \mathbb{Z}} G^\pm(n + \frac{1}{2})z^{-n-2},$$

$$(5.25) \quad \bar{G}^\pm(z) = Y(\bar{\tau}^\pm, z) = \sum_{n \in \mathbb{Z}} \bar{G}^\pm(n + \frac{1}{2})z^{-n-2},$$

$$(5.26) \quad L(z) = Y(\omega, z) = \sum_{n \in \mathbb{Z}} L(n)z^{-n-2}.$$

Lemma 5.1. *The components of the fields (5.23)- (5.26) satisfy the commutation relations of the $N = 4$ superconformal algebra with central charge $c = -9$.*

Proof. Standard calculation in lattice vertex algebras implies

$$\tau_0^+ \bar{\tau}^\pm = J^\pm(-2)\mathbf{1}, \quad \tau_1^+ \bar{\tau}^\pm = 2J^\pm(-1)\mathbf{1}, \quad \tau_n^\pm \bar{\tau}^\pm = 0 \quad (n \geq 2).$$

Moreover

$$(5.27) \quad \tau^+, \bar{\tau}^-, j^0, \omega$$

$$(5.28) \quad \tau^-, \bar{\tau}^+, -j^0, \omega$$

are $N = 2$ superconformal vectors, i.e., they generate free-field realizations of $N = 2$ superconformal algebra with $c = -9$ inside of $\beta\gamma bc$ system. By using relation (4.16) and the fact that Q

is a screening operator we have that $\tau^+, \bar{\tau}^+$ are highest weight vector for the affine Lie algebra $\widehat{sl_2}$ generated by $J^\pm(z)$ and $J^0(z)$ and

$$U(\widehat{sl_2}).\tau^+ \cong U(\widehat{sl_2}).\bar{\tau}^+ \cong L_{A_1}(-\frac{5}{2}\Lambda_0 + \Lambda_1)$$

In particular $U(sl_2).\tau^+$ (resp. $U(sl_2).\bar{\tau}^+$) is two-dimensional spanned by τ^\pm (resp. $\bar{\tau}^\pm$). From the arguments described above and using commutator formula, we get the assertion. \square

Let us denote the $N = 4$ superconformal algebra with \mathcal{A} . Consider

$$V = U(\mathcal{A}).\mathbf{1}.$$

Then V is a highest weight \mathcal{A} -module. Moreover, V is a $\frac{1}{2}\mathbb{Z}_{\geq 0}$ -graded vertex operator superalgebra with central charge $c = -9$.

So we have:

Proposition 5.1. *V is a subalgebra of the Clifford-Weyl vertex superalgebra $M \otimes F$.*

Lemma 5.2. *We have:*

$$G^+(-\frac{3}{2})\bar{G}^+(-\frac{3}{2})\mathbf{1} = -2e(-1)\omega + h(-1)e(-2) - h(-2)e(-1).$$

Proof. By using our realization and standard calculation in lattice vertex superalgebras we get

$$\begin{aligned} G^+(-\frac{3}{2})\bar{G}^+(-\frac{3}{2})\mathbf{1} &= e_{-1}^\delta(\alpha(-1) + \beta(-1) - 2\delta(-1))e^{\alpha+\beta-\delta} \\ (5.29) \quad &= ((\alpha(-1) + \beta(-1))\delta(-1) - \delta(-1)^2 + \delta(-2))e^{\alpha+\beta}. \end{aligned}$$

By using (4.15) we get

$$\begin{aligned} &G^+(-\frac{3}{2})\bar{G}^+(-\frac{3}{2})\mathbf{1} + 2e(-1)\omega \\ &= ((\alpha(-1) + \beta(-1))(-2\beta(-1) + \delta(-1)))e^{\alpha+\beta} + (2\beta(-2) - \delta(-2))e^{\alpha+\beta} \\ &= h(-1)e(-2) - h(-2)e(-1), \end{aligned}$$

and the lemma follows. \square

6. IRREDUCIBLE V -MODULES IN THE CATEGORY \mathcal{O}

In this section we shall classify irreducible V -modules which are in the category \mathcal{O} as modules for the $N = 4$ superconformal algebra. This classification reduces to the classification of all irreducible highest weight modules for the Zhu's algebra.

So we shall first identify Zhu's algebra of V .

Proposition 6.1.

- (i) *Zhu's algebra $A(V)$ is isomorphic to a certain quotient of $U(sl_2)$.*
- (ii) *In Zhu's algebra $A(V)$ we have the following relation:*

$$[e]([\omega] + \frac{1}{2}) = 0.$$

Proof. Since V is strongly generated by $\tau^\pm, \bar{\tau}^\pm, e, h, f$ we have that Zhu's algebra $A(V)$ is generated by

$$[\tau^\pm], [\bar{\tau}^\pm], [e], [f], [h].$$

From the definition of Zhu's algebra for vertex operator superalgebras follows that

$$\tau^\pm, \bar{\tau}^\pm \in O(V)$$

and therefore $[\tau^\pm] = [\bar{\tau}^\pm] = 0$. This implies that $A(V)$ is generated by $[e], [f], [h]$ which satisfy commutation relations for sl_2 . This proves assertion (i). (Note that Zhu's algebra of $N_{A_1}(k\Lambda_0)$ is isomorphic to $U(sl_2)$.)

Now we shall prove relation (ii). By using Lemma 5.2 we get that in $A(V)$ we have

$$(6.30) \quad [G^+(-\frac{3}{2})\bar{G}^+(-\frac{3}{2})\mathbf{1}] = -2[e][\omega].$$

By definition of Zhu's algebra we have

$$(6.31) \quad [G^+(-\frac{3}{2})\bar{G}^+(-\frac{3}{2})\mathbf{1}] + [G^+(-\frac{1}{2})\bar{G}^+(-\frac{3}{2})\mathbf{1}] = 0.$$

Since $[G^+(-\frac{1}{2})\bar{G}^+(-\frac{3}{2})\mathbf{1}] = [e(-2)\mathbf{1}] = -[e]$, relations (6.30) and (6.31) gives that

$$-2[e][\omega] - [e] = 0,$$

which proves the proposition. \square

The previous proposition and Zhu's algebra theory imply that every irreducible $\frac{1}{2}\mathbb{Z}_{\geq 0}$ -graded V -module has the form $L_c^{N=4}(U)$, where U is a certain irreducible $U(sl_2)$ -module annihilated by $[e](\omega + 1/2)$ and $L_c^{N=4}(U)$ is a $\frac{1}{2}\mathbb{Z}_{\geq 0}$ -graded V -module

$$L_c^{N=4}(U) = \bigoplus_{m \in \frac{1}{2}\mathbb{Z}_{\geq 0}} L_c^{N=4}(U)(m), \quad L_c^{N=4}(U)(0) \cong U.$$

The classification of irreducible V -modules in the category \mathcal{O} is then equivalent to the classification of irreducible, highest weight modules for Zhu's algebra $A(V)$.

Proposition 6.2. *Let $r \in \mathbb{C}$ and U_r be the irreducible highest weight sl_2 -module with highest weight $r\omega_1$. Then U_0 and U_{-1} are the only irreducible, highest weight $A(V)$ -modules. Therefore, $L_c^{N=4}(U_0)$ and $L_c^{N=4}(U_{-1})$ give a complete list of irreducible V -modules from the category \mathcal{O} .*

Proof. Let $U = U_r$ be an irreducible, highest weight module for $A(V)$ and $L_c^{N=4}(U)$ the associated V -module. Then

$$L(0)u = [\omega].u = \frac{r(r+2)}{2}u \quad \forall u \in U.$$

This implies that

$$[e](\omega + 1/2)U = 0 \implies U = 0 \quad \text{or} \quad U = U_{-1}.$$

Clearly, U_0 is an $A(V)$ -module since it is lowest component of V . U_{-1} is also a module for Zhu's algebra since the lowest component of $M \otimes F$ is $(M \otimes F)(0) = U(sl_2).e^{-\delta} \cong U_{-1}$. The proof follows. \square

Theorem 6.1. *V is a simple vertex operator superalgebra, i.e., $V \cong L_c^{N=4}$ with $c = -9$.*

Proof. Assume that V is not simple. Then V contains an graded ideal $I \subset V$, $0 \neq I \neq V$ such that

$$I = \bigoplus_{n \in \frac{1}{2}\mathbb{Z}_{\geq 0}} I(n + n_0), \quad L(0)|I(r) \cong r\text{Id}, \quad I(n_0) \neq 0.$$

Since $I \neq V$, we have that $n_0 > 0$. Then the lowest component $I(n_0)$ has to be a module for the Zhu's algebra $A(V)$. Proposition 6.1 then implies that either $e(0)I(n_0) = 0$ or $L(0)|I(n_0) = -\frac{1}{2}\text{Id}$. But both cases lead to a contradiction. \square

We notice that $\delta(0)$ acts semi-simply on V and it defines the \mathbb{Z} -graduation:

$$V = \bigoplus_{\ell \in \mathbb{Z}} V^{(\ell)}, \quad V^{(\ell)} = \{v \in V \mid \delta(0)v = \ell v\}.$$

The simplicity of V and the fact that

$$(6.32) \quad V^{(\ell_1)} \cdot V^{(\ell_2)} \subset V^{(\ell_1 + \ell_2)} \quad \forall \ell_1, \ell_2 \in \mathbb{Z}$$

implies:

Corollary 6.1. $V^{(0)}$ is a simple vertex algebra of central charge $c = -9$ and each $V^{(\ell)}$ is a simple $V^{(0)}$ -module.

Corollary 6.2. Let $c = -9$. We have the following isomorphisms of vertex superalgebras:

$$L_c^{N=4} \cong (M \otimes F)^{int} \cong \text{Ker}_{M \otimes F} \tilde{Q}.$$

Proof. Since all generators of V belong to the vertex superalgebra $(M \otimes F)^{int}$, we have that V is its subalgebra. As in the previous theorem we conclude that $(M \otimes F)^{int}$ must be simple as V -module. Similarly we prove that $V \cong \text{Ker}_{M \otimes F} \tilde{Q}$. \square

Let:

$$SC\Lambda(1) = (M \otimes F)^{int} = V, \quad SC\Pi(1) = (M \otimes F)/SC\Lambda(1).$$

Proposition 6.3.

- (1) $SC\Pi(1)$ is an irreducible $U(\mathcal{A})$ -module from the category \mathcal{O} .
- (2) $SC\Lambda(1)$ and $SC\Pi(1)$ give all non-isomorphic V -modules from the category \mathcal{O} .
- (3) $M \otimes F$ is an indecomposable V -module.

Proof. Proof of assertion (i) uses the structure of Zhu's algebra $A(V)$ and it is similar to that of Theorem 6.1. Assertion (ii) follows from Proposition 6.2, and (iii) from the fact that $M \otimes F$ is not semi-simple V -module. \square

7. IRREDUCIBLE V -MODULES OUTSIDE OF THE CATEGORY \mathcal{O}

We have seen in the previous section that V has only two irreducible modules in the category \mathcal{O} . But modules from the category \mathcal{O} don't form a semisimple category. In this section we shall classify all irreducible, $1/2\mathbb{Z}_{\geq 0}$ -graded V -modules in the category of weight modules.

We say that a $U(\mathcal{A})$ -module M is a weight module if $L(0)$ and $h(0)$ act semisimply on M .

For every $(\mu, r) \in \mathbb{C}$ we define the following $U(sl_2)$ -module $U_{\mu, r}$. As a vector space

$$U_{\mu, r} = \text{span}_{\mathbb{C}}\{E_i, i \in \mathbb{Z}\},$$

the sl_2 action is defined by

$$\begin{aligned} eE_i &= E_{i-1} \\ hE_i &= (-2r - 2i + \mu)E_i \\ fE_i &= -(r + i + 1)(r + i - \mu)E_{i+1}. \end{aligned}$$

Let $\Omega = ef + fe + \frac{1}{2}h^2$ be the Casimir element of $U(sl_2)$. Then $\Omega w = \frac{\mu(\mu+2)}{2}w$ for every $w \in U_{\mu, r}$.

Lemma 7.1. *Assume that U is an irreducible, weight $A(V)$ -module. Then U is isomorphic to exactly one of the modules from the list*

$$U_0, U_{-1}, U_{-1}^*, U_{-1, r} \quad (r \in \mathbb{C} \setminus \mathbb{Z}).$$

Now we want to show that $U_{-1, r}$ are modules for Zhu's algebra $A(V)$ and to construct the associated simple V -modules $L_c^{N=4}(U_{-1, r})$.

First we notice that for every $\lambda \in \mathbb{C}$

$$V_{L+\lambda(\alpha+\beta)} := V_L \cdot e^{\lambda(\alpha+\beta)}$$

is an σ_λ -twisted V_L -module, where $\sigma_\lambda := \exp[\lambda(\alpha + \beta)(0)]$ is V_L -automorphism. This twisted module can be constructed as

$$(V_{L+\lambda(\alpha+\beta)}, \tilde{Y}(\cdot, z)) := (V_L, Y_{V_L}(\Delta(\lambda(\alpha + \beta))(-1)\mathbf{1}, z) \cdot, z).$$

Since $\sigma_\lambda \equiv \text{Id}$ on $M \otimes F$ and V , we conclude that $V_{L+\lambda(\alpha+\beta)}$ is an untwisted V -module. Take $\lambda = -r - 1$ and consider V -submodule

$$(7.33) \quad M(r) := \mathbb{C}[\beta + (\mathbb{Z} + \lambda)(\alpha + \beta)] \otimes M_{\alpha, \beta}(1) \otimes F$$

where $M_{\alpha, \beta}(1)$ is the Heisenberg subalgebra of V_L generated by $\alpha(z)$ and $\beta(z)$.

Theorem 7.1. *Assume that $r \notin \mathbb{Z}$. Then $M(r)$ is an irreducible $\frac{1}{2}\mathbb{Z}_{\geq 0}$ -graded V -module whose lowest component is*

$$M(r)(0) := U(sl_2) \cdot e^{\beta - \delta - (r+1)(\alpha+\beta)} \cong U_{-1, r}.$$

In particular, $M(r) \cong L_c^{N=4}(U_{-1, r})$.

Proof. It is clear that the lowest component of $M(r)$ is

$$M(r)(0) = \text{span}_{\mathbb{C}}\{E_i = e^{\beta - \delta - (r+1+i)(\alpha+\beta)}, i \in \mathbb{Z}\},$$

and that $L(0)E_i = -\frac{1}{2}E_i$, $i \in \mathbb{Z}$. By using explicit formula for e, h, f (4.7)-(4.9) we see that $M(r)(0) \cong U_{-1, r}$ as $U(sl_2)$ -modules and as $A(V)$ -modules. Assume now that $M(r)$ is not irreducible. Then it must contain a graded submodule which is not generated by vectors from $M(r)(0)$. By using information about Zhu's algebra $A(V)$, we conclude that then $M(r)$ should contain vector w such that $L(0)w = 0$, $ew = fw = hw = 0$. But vectors of conformal weight 0 are in the linear span of

$$\{e^{\beta - (r+1+i)(\alpha+\beta)}, e^{\beta - (r+1+i)(\alpha+\beta) - 2\delta} \mid i \in \mathbb{Z}\},$$

which can not generate a V -submodule with lowest conformal weight 0. The proof follows. \square

For any V -module M such that $L(0)$ and $h(0)$ act semi-simply with finite-dimensional (common) eigenspaces we define $\text{ch}_M(q, z) = \text{tr} q^{L(0)} z^{h(0)}$. By using (7.33) and properties of δ -function one show the following result.

Proposition 7.1. *We have*

$$\text{ch}_{M(r)}(q, z) = z^{-2r} \delta(z^2) \prod_{n=1}^{\infty} (1 - q^n)^{-2} \prod_{n=1}^{\infty} (1 + q^{n-3/2} z^{-1})(1 + q^{n+1/2} z).$$

8. LOGARITHMIC V -MODULES

Now we shall apply the general method for constructing logarithmic modules developed in [AM3] to construct a series of logarithmic modules for the vertex superalgebra V .

First we notice that

$$(8.34) \quad \Pi(0) = \mathbb{C}[\mathbb{Z}(\alpha + \beta)] \otimes M_{\alpha, \beta}(1)$$

is a vertex subalgebra of V_L and that for every $\lambda \in \mathbb{C}$

$$\Pi(\lambda) = \mathbb{C}[(\mathbb{Z} + \lambda)(\alpha + \beta)] \otimes M_{\alpha, \beta}(1) = \Pi(0) \cdot e^{\lambda(\alpha + \beta)}$$

is its irreducible module.

Remark 8.1. *This vertex algebra also appeared in [A3], [BDT] and [Fr].*

Now define the following vertex superalgebra

$$S\Pi(0) = \Pi(0) \otimes F \subset V_L$$

and its irreducible modules

$$S\Pi(\lambda) = S\Pi(\lambda) \otimes F \subset V_{L+\lambda(\alpha+\beta)}.$$

We note also

$$V \subset M \otimes F \subset S\Pi(0).$$

Consider now the extended vertex superalgebra (in the sense of [AM3])

$$S\mathcal{V} := S\Pi(0) \oplus S\Pi(-1/2)$$

and its modules

$$S\mathcal{V}(\lambda) := S\Pi(\lambda) \oplus S\Pi(\lambda - 1/2).$$

Let

$$v = e^{-\frac{1}{2}(\alpha+\beta)+\delta} \in S\Pi(-1/2).$$

and recall that $\tilde{Q} = \text{Res}_z Y(v, z)$ is a screening operator which commutes with the action of $U(\mathcal{A})$.

By using results of [AM3] we get:

Theorem 8.1. *For every $\lambda \in \mathbb{C}$*

$$(8.35) \quad (\widetilde{\mathcal{SV}}(\lambda), \widetilde{Y}_{\widetilde{\mathcal{SV}}(\lambda)}(\cdot, z)) := (\mathcal{SV}(\lambda), Y_{\mathcal{SV}(\lambda)}(\Delta(v, z)\cdot, z))$$

is a logarithmic V -module. The action of the Virasoro algebra is

$$\widetilde{L}(z) = \sum_{n \in \mathbb{Z}} \widetilde{L}(n) z^{-n-2} = \widetilde{Y}_{\widetilde{\mathcal{SV}}(\lambda)}(\omega, z) = L(z) + z^{-1} Y_{\mathcal{SV}(\lambda)}(v, z).$$

In particular,

$$\widetilde{L}(0) = L(0) + \widetilde{Q}$$

and $\widetilde{\mathcal{SV}}(\lambda)$ has $\widetilde{L}(0)$ -nilpotent rank two.

Let us describe these logarithmic modules in more detail.

We have:

$$\begin{aligned} \Delta(v, z)e &= e, \\ \Delta(v, z)h &= h, \\ \Delta(v, z)f &= f + \frac{1}{2}z^{-1}e^{-\frac{3}{2}(\alpha+\beta)+\delta}, \\ \Delta(v, z)\tau^\pm &= \tau^\pm, \\ \Delta(v, z)\bar{\tau}^+ &= \bar{\tau}^+ + 2z^{-1}e^{\frac{1}{2}(\alpha+\beta)}, \\ \Delta(v, z)\bar{\tau}^- &= \bar{\tau}^- - 2z^{-1}D(e^{-\frac{1}{2}(\alpha+\beta)}) + z^{-2}e^{-\frac{1}{2}(\alpha+\beta)}. \end{aligned}$$

The formulas above and (8.35) completely determine the action of the $N = 4$ superconformal algebra on logarithmic modules.

The following corollary shows that logarithmic V -modules appear in the extension of non-logarithmic (weight) V -modules:

Corollary 8.1. *The logarithmic V -module $\widetilde{\mathcal{SV}}(\lambda)$ appears in the extension*

$$0 \rightarrow S\Pi(\lambda - 1/2) \rightarrow \widetilde{\mathcal{SV}}(\lambda) \rightarrow S\Pi(\lambda) \rightarrow 0.$$

of non-logarithmic weak V -modules $S\Pi(\lambda - 1/2)$ and $S\Pi(\lambda)$.

9. TWISTED V -MODULES

Let $\mu \in \mathbb{R}$, $0 \leq \mu < 1$ and consider now the automorphism $g_\mu = e^{2\pi i \mu \delta(0)}$ of $M \otimes F$. One can easily see that V is g_μ -invariant and that g_μ -acts semisimply on V .

We have the following result on the structure of twisted Zhu's algebra $A_{g_\mu}(V)$.

Proposition 9.1. *Assume that $\mu + \mathbb{Z} \neq \frac{1}{2} + \mathbb{Z}$.*

- (i) *Zhu's algebra $A_{g_\mu}(V)$ is isomorphic to a certain quotient of $U(sl_2)$.*
- (ii) *In Zhu's algebra $A_{g_\mu}(V)$ we have the following relation:*

$$[e]([\omega] + \frac{(1+\mu)(1-\mu)}{2}) = 0.$$

Proof. Proof is similar to the untwisted case. Since $A_{g_\mu}(V)$ is generated by

$$[\tau^\pm], [\bar{\tau}^\pm], [e], [f], [h],$$

and since

$$\tau^\pm, \bar{\tau}^\pm \in O_{g_\mu}(V),$$

we conclude that $A_{g_\mu}(V)$ is again generated by $[e], [f], [h]$ which satisfy commutation relations for sl_2 . This proves assertion (i).

Definition of twisted Zhu's algebra $A_{g_\mu}(V)$ implies that

$$(9.36) \quad \begin{aligned} & [G^+(-\frac{3}{2})\bar{G}^+(-\frac{3}{2})\mathbf{1}] + (1+\mu)[G^+(-\frac{1}{2})\bar{G}^+(-\frac{3}{2})\mathbf{1}] + \\ & \binom{1+\mu}{2}[G^+(\frac{1}{2})\bar{G}^+(-\frac{3}{2})\mathbf{1}] = 0. \end{aligned}$$

Now assertion (ii) follows from Lemma 5.2. \square

First we will consider g_μ -twisted V -module $M \otimes F^\mu$. Recall that this module is a direct sum of Wakimoto modules for $\widehat{sl_2}$.

We have the following irreducibility result:

Proposition 9.2. *Assume that $\mu \notin \frac{1}{2}\mathbb{Z}$. Then $M \otimes F^\mu$ is an irreducible V -module.*

Proof. $M \otimes F^\mu$ is $\frac{1}{2}\mathbb{Z}_{\geq 0}$ -graded and its lowest component is generated by the vector $1 \otimes e^{(\mu-1)\delta}$. This vector is a highest weight vector for sl_2 such that

$$h(0)1 \otimes e^{(\mu-1)\delta} = (\mu-1)(1 \otimes e^{(\mu-1)\delta}).$$

So, lowest component is an irreducible sl_2 -module. Irreducibility of $M \otimes F^\mu$ follows easily by using structure of Zhu's algebra $A_{g_\mu}(V)$. Proof is similar to that of untwisted case. \square

Next we consider a family of g_μ -twisted V -modules:

$$(M^\mu(r), \tilde{Y}_{M^\mu(r)}(\cdot, z) := (M^\mu, Y_{M(r)}(\Delta(\mu\delta(-1)\mathbf{1})\cdot, z)).$$

Then $M^\mu(r)$ is g_μ -twisted V -module. $M^\mu(r)$ can be realized as a submodule of

$$V_L \cdot e^{\lambda(\alpha+\beta)+\mu\delta}$$

where $\lambda = -r - 1$. In fact:

$$(9.37) \quad M^\mu(r) := \mathbb{C}[\beta + (\mathbb{Z} + \lambda)(\alpha + \beta)] \otimes M_{\alpha, \beta}(1) \otimes F^\mu,$$

where F^μ is a g_μ -twisted F -module realized on $V_{\mathbb{Z}\delta} \cdot e^{\mu\delta}$.

Theorem 9.1. *Assume that $r \notin \mathbb{Z}$, $\mu \notin \frac{1}{2}\mathbb{Z}$ and $r - \mu \notin \mathbb{Z}$. Then $M^\mu(r)$ is an irreducible g_μ -twisted $\frac{1}{2}\mathbb{Z}_{\geq 0}$ -graded V -module whose lowest component is*

$$M^\mu(r)(0) := U(sl_2) \cdot e^{\beta+(\mu-1)\delta-(r+1)(\alpha+\beta)} \cong U_{\mu-1, r}.$$

Proof. The proof uses the irreducibility of lowest component and the structure of twisted Zhu's algebra $A_{g_\mu}(V)$ from Proposition 9.1. \square

Fix the following \mathbb{Z} -graduation on $M^\mu(s)$:

$$M^\mu(s) = \bigoplus_{j \in \mathbb{Z}} (M^\mu(s))^j \quad (M^\mu(s))^j = \{v \in M^\mu(s) \mid \delta(0)v = (j + \mu - 1)v\}.$$

10. REALIZATION OF THE SIMPLE VERTEX OPERATOR ALGEBRA $L_{A_2}(-\frac{3}{2}\Lambda_0)$

Let \mathfrak{g} be the simple, complex Lie algebra of type A_2 . The root system of \mathfrak{g} is given by

$$\Delta = \{\varepsilon_i - \varepsilon_j \mid 1 \leq i, j \leq 3 \ i \neq j\}$$

with $\alpha_1 = \varepsilon_1 - \varepsilon_2$, $\alpha_2 = \varepsilon_2 - \varepsilon_3$ being simple roots. The highest root is $\theta = \varepsilon_1 - \varepsilon_3$. We shall fix root vectors and coroots as usual. For any positive root $\alpha \in \Delta_+$ denote by e_α and f_α the root vectors corresponding to α and $-\alpha$, respectively. Let h_α be the corresponding coroot.

Let $\hat{\mathfrak{g}}$ be the affine Lie algebra of type $A_2^{(1)}$. Denote by Λ_i , $i = 0, 1, 2$ its fundamental weights. Let $L_{A_2}(k\Lambda_0)$ be the simple vertex (operator) algebra of level k associated to $\hat{\mathfrak{g}}$.

Now we shall prove that the simple affine vertex operator algebra $L_{A_2}(-\frac{3}{2}\Lambda_0)$ can be embedded into $V \otimes F_{-1}$, where F_{-1} is the simple lattice vertex superalgebra $V_{\mathbb{Z}\varphi}$ associated to the lattice $\mathbb{Z}\varphi$ where $\langle \varphi, \varphi \rangle = -1$. As a vector space, $F_{-1} = M_\varphi(1) \otimes \mathbb{C}[\mathbb{Z}\varphi]$ where $M_\varphi(1)$ is the Heisenberg vertex algebra generated by the field $\varphi(z) = \sum_{n \in \mathbb{Z}} \varphi(n)z^{-n-1}$, and $\mathbb{C}[\mathbb{Z}\varphi]$ is the group algebra with generator e^φ . F_{-1} admits the following \mathbb{Z} -graduation:

$$F_{-1} = \bigoplus_{m \in \mathbb{Z}} F_{-1}^{(m)}, \quad F_{-1}^{(m)} = M_\varphi(1) \otimes e^{m\varphi}.$$

Define

$$(10.38) \quad e_\theta := J^+ \otimes \mathbf{1} = e \otimes \mathbf{1}$$

$$(10.39) \quad f_\theta := J^- \otimes \mathbf{1} = f \otimes \mathbf{1}$$

$$(10.40) \quad e_{\alpha_1} := \frac{1}{\sqrt{2}} \tau^+ \otimes e^\varphi$$

$$(10.41) \quad f_{\alpha_1} := \frac{1}{\sqrt{2}} \bar{\tau}^- \otimes e^{-\varphi}$$

$$(10.42) \quad e_{\alpha_2} := \frac{1}{\sqrt{2}} \bar{\tau}^+ \otimes e^{-\varphi}$$

$$(10.43) \quad f_{\alpha_2} := \frac{1}{\sqrt{2}} \tau^- \otimes e^\varphi$$

$$(10.44) \quad h_{\alpha_1} := (-\beta - \frac{3}{2}\varphi + \frac{1}{2}\delta)(-1)$$

$$(10.45) \quad h_{\alpha_2} := (-\beta + \frac{3}{2}\varphi + \frac{1}{2}\delta)(-1).$$

Lemma 10.1. *Formulas (10.38)-(10.45) define a vertex algebra homomorphism:*

$$\Phi : N_{A_2}(-\frac{3}{2}\Lambda_0) \rightarrow V \otimes F.$$

Proof. Since $\tau^+, \bar{\tau}^-$ generate $N = 2$ superconformal vertex algebra with $c = -9$, then $e_{\alpha_1}, f_{\alpha_1}, h_{\alpha_1}$ must generate affine vertex algebra $A_1^{(1)}$ at level $k = -3/2$, i.e., $L_{A_1}(-\frac{3}{2}\Lambda_0)$ (see [A1] and [FST]). Similarly $e_{\alpha_2}, f_{\alpha_2}, h_{\alpha_2}$ also generate a copy of $L_{A_1}(-\frac{3}{2}\Lambda_0)$. By using defining relations for the $N = 4$ superconformal algebra we have:

$$\begin{aligned} e_{\alpha_1}(0)f_{\alpha_2} &= e_{\alpha_2}(0)f_{\alpha_1} = 0, \\ e_{\alpha_1}(0)e_{\alpha_2} &= \frac{1}{2}(G^+(1/2)\bar{G}^+(-3/2)\mathbf{1}) \otimes \mathbf{1} = e \otimes \mathbf{1} = e_{\theta}, \\ f_{\alpha_1}(0)f_{\alpha_2} &= \frac{1}{2}(\bar{G}^-(1/2)G^+(-3/2)\mathbf{1}) \otimes \mathbf{1} = -f \otimes \mathbf{1} = -f_{\theta}, \\ e(0)f_{\alpha_1} &= \frac{1}{\sqrt{2}}(e(0)\bar{\tau}^-) \otimes e^{-\varphi} = -\frac{1}{\sqrt{2}}\bar{\tau}^+ \otimes e^{-\varphi} = -e_{\alpha_2}, \\ e(0)f_{\alpha_2} &= \frac{1}{\sqrt{2}}(e(0)\tau^-) \otimes e^{\varphi} = \frac{1}{\sqrt{2}}\tau^+ \otimes e^{\varphi} = e_{\alpha_1}, \\ f(0)e_{\alpha_1} &= \frac{1}{\sqrt{2}}(f(0)\tau^+) \otimes e^{\varphi} = \frac{1}{\sqrt{2}}\bar{\tau}^- \otimes e^{\varphi} = f_{\alpha_2}, \\ f(0)e_{\alpha_2} &= \frac{1}{\sqrt{2}}(f(0)\bar{\tau}^-) \otimes e^{-\varphi} = -\frac{1}{\sqrt{2}}\bar{\tau}^+ \otimes e^{-\varphi} = -f_{\alpha_1}. \end{aligned}$$

Now assertion follows from the commutator formula for vertex algebras. \square

Denote by $\pi_s, s \in \mathbb{Z}$, the automorphism of $U(\hat{\mathfrak{g}})$ uniquely determined by

$$\begin{aligned} \pi_s(e_{\alpha_1}(n)) &= e_{\alpha_1}(n+s), \pi_s(e_{\alpha_2}(n)) = e_{\alpha_2}(n-s) \\ \pi_s(f_{\alpha_1}(n)) &= f_{\alpha_1}(n-s), \pi_s(f_{\alpha_2}(n)) = f_{\alpha_2}(n+s) \\ \pi_s(h_{\alpha_1}(n)) &= h_{\alpha_1}(n) + sk\delta_{n,0}, \pi_s(h_{\alpha_2}(n)) = h_{\alpha_2}(n) - sk\delta_{n,0}. \end{aligned}$$

In our explicit realization operator $e^{s\varphi}$ is such automorphism. We have:

Theorem 10.1. *The subalgebra of $V \otimes F_{-1}$ generated by vectors (10.38)-(10.45) is isomorphic to the vertex operator algebra $L_{A_2}(-\frac{3}{2}\Lambda_0)$. Moreover, $V \otimes F_{-1}$ is a simple current extension of $L_{A_2}(-\frac{3}{2}\Lambda_0)$ and*

$$V \otimes F_{-1} = \bigoplus_{s \in \mathbb{Z}} \pi_s(L_{A_2}(-\frac{3}{2}\Lambda_0))$$

where $\pi_s(L_{A_2}(-\frac{3}{2}\Lambda_0)) = e^{s\varphi}L_{A_2}(-\frac{3}{2}\Lambda_0)$.

Proof. First we notice that $(\delta + \varphi)(0)$ acts semi-simply on the simple vertex superalgebra $V \otimes F_{-1}$ and defines the following \mathbb{Z} -graduation

$$V \otimes F_{-1} = \bigoplus_{\ell} (V \otimes F_{-1})^{(\ell)},$$

where

$$(V \otimes F_{-1})^{(\ell)} = \{v \in V \otimes F_{-1} \mid (\delta + \varphi)(0)v = \ell v\}.$$

Then $(V \otimes F_{-1})^{(0)}$ is a simple vertex algebra and each $(V \otimes F_{-1})^{(\ell)}$ is a simple $(V \otimes F_{-1})^{(0)}$ -module.

For the proof of simplicity, it suffices to prove that $(V \otimes F_{-1})^{(0)}$ is generated by vectors (10.38)-(10.45). This can be proved by using relations

$$\begin{aligned}\tau^+ \otimes \mathbf{1} &= \sqrt{2}e_{\alpha_1}(-2).(\mathbf{1} \otimes e^{-\varphi}), \\ \tau^- \otimes \mathbf{1} &= \sqrt{2}f_{\alpha_2}(-2).(\mathbf{1} \otimes e^{-\varphi}), \\ \bar{\tau}^+ \otimes \mathbf{1} &= \sqrt{2}e_{\alpha_2}(-2).(\mathbf{1} \otimes e^{\varphi}), \\ \bar{\tau}^- \otimes \mathbf{1} &= \sqrt{2}f_{\alpha_1}(-2).(\mathbf{1} \otimes e^{\varphi}).\end{aligned}$$

and analogous proof to that of Theorem 6.1 from [A3]. \square

Corollary 10.1. *We have the following inclusions of vertex algebras*

$$L_{A_2}(-\frac{3}{2}\Lambda_0) \subset M \otimes \tilde{\Pi}(0) \subset \Pi(0) \otimes \tilde{\Pi}(0)$$

where $\Pi(0)$ is defined by (8.34) and

$$\tilde{\Pi}(0) = \mathbb{C}[\mathbb{Z}(\delta + \varphi)] \otimes M_{\delta, \varphi}(1).$$

Let $\mu \in \mathbb{R}$. Consider now $h_\mu := e^{2\pi i \mu \varphi(0)}$ twisted F_{-1} -module

$$F_{-1}^\mu := V_{\mathbb{Z}\varphi}.e^{\mu\varphi}.$$

As a $M_\varphi(1)$ -module:

$$F_{-1}^\mu = \bigoplus_{j \in \mathbb{Z}} F_{-1}^{\mu+j}, \quad F_{-1}^{\mu+j} = M_\varphi(1).e^{(j+\mu)\varphi}.$$

Let U be any $g_\mu := e^{2\pi i \mu \delta(0)}$ -twisted V -module. Since $g_\mu h_\mu = e^{2\pi i \mu (\delta + \varphi)(0)}$ acts trivially on $L_{A_2}(-\frac{3}{2}\Lambda_0) = (V \otimes F_{-1})^0$, we conclude that $U \otimes F_{-1}^\mu$ is an untwisted $L_{A_2}(-\frac{3}{2}\Lambda_0)$ -module. The proof of the following result is analogous to that of Theorem 6.2 from [A3].

Theorem 10.2. *Assume that U is a g_μ -twisted V -module such that U admits the following \mathbb{Z} -graduation*

$$(10.46) \quad U = \bigoplus_{j \in \mathbb{Z}} U^{(j)}, \quad V^{(i)}.U^{(j)} \subset U^{(i+j)}.$$

Then

$$U \otimes F_{-1}^\mu = \bigoplus_{s \in \mathbb{Z}} \mathcal{L}_s(U), \quad \text{where } \mathcal{L}_s(U) := \bigoplus_{i \in \mathbb{Z}} U^i \otimes F_{-1}^{-s+i+\mu},$$

is an $L_{A_2}(-\frac{3}{2}\Lambda_0)$ -module.

If U is irreducible g_μ -twisted V -module, then for every $s \in \mathbb{Z}$ $\mathcal{L}_s(U)$ is an irreducible (untwisted) $L_{A_2}(-\frac{3}{2}\Lambda_0)$ -module.

Irreducible modules in the category \mathcal{O} were classified in [P]. This vertex operator algebra is rational in the category \mathcal{O} (cf. [AM1], [Ar2]). Here we have free field realization of these modules:

Corollary 10.2. *All irreducible $L_{A_2}(-\frac{3}{2}\Lambda_0)$ -modules in the category \mathcal{O} can be realized as follows:*

$$\begin{aligned} L_{A_2}(-\frac{3}{2}\Lambda_0) &\cong U(\hat{\mathfrak{g}}).\mathbf{1}, & L_{A_2}(-\frac{1}{2}\Lambda_0 - \frac{1}{2}\Lambda_1 - \frac{1}{2}\Lambda_2) &\cong U(\hat{\mathfrak{g}}).e^{-\delta}, \\ L_{A_2}(-\frac{3}{2}\Lambda_1) &\cong U(\hat{\mathfrak{g}}).e^{-\beta+\frac{1}{2}\delta-\frac{1}{2}\varphi}, & L_{A_2}(-\frac{3}{2}\Lambda_2) &\cong U(\hat{\mathfrak{g}}).e^{-\beta+\frac{1}{2}\delta+\frac{1}{2}\varphi}. \end{aligned}$$

Using our realization one easily see that the category \mathcal{O} is not closed under fusion.

Now we will apply Theorem 10.2 to irreducible untwisted V -modules from Section 7 and twisted V -modules constructed in Section 9.

Corollary 10.3. *Assume that $0 < \mu < 1$, $\mu \neq 1/2$.*

- (1) *For every $r \in \mathbb{C} \setminus \mathbb{Z}$ and every $s \in \mathbb{Z}$, $\mathcal{L}_s(M(r))$ is an irreducible $L_{A_2}(-\frac{3}{2}\Lambda_0)$ -module.*
- (2) *For every $s \in \mathbb{Z}$, $\mathcal{L}_s(M \otimes F^\mu)$ is an irreducible $L_{A_2}(-\frac{3}{2}\Lambda_0)$ -module.*
- (3) *For every $r \in \mathbb{C} \setminus \mathbb{Z}$ such that $r - \mu \notin \mathbb{Z}$ and every $s \in \mathbb{Z}$, $\mathcal{L}_s(M^\mu(r))$ is an irreducible $L_{A_2}(-\frac{3}{2}\Lambda_0)$ -module.*

By construction, the Sugawara Virasoro vector in $L_{A_2}(-\frac{3}{2}\Lambda_0)$ is

$$(10.47) \quad \omega_{A_2} = \omega - \frac{1}{2}\varphi(-1)^2\mathbf{1}.$$

So in Zhu's algebra we have $[\omega_{A_2}] = [\omega] - 1/2[\varphi(-1)\mathbf{1}]^2$. Using this relation and $[e_{\alpha_1} \circ e_{\alpha_2}] = 0$, we get the following relation in Zhu's algebra $A(L_{A_2}(-\frac{3}{2}\Lambda_0))$:

$$(10.48) \quad [e]([\omega_{A_2}] + 1/2) = 0.$$

(See also [P] for analysis of Zhu's algebra of this vertex algebra). So we have analogous relations in Zhu's algebras $A(V)$ and $A(L_{A_2}(-\frac{3}{2}\Lambda_0))$.

Among irreducible modules constructed in Corollary 10.3 we have a family of $\mathbb{Z}_{\geq 0}$ -graded modules realized on the irreducible $\Pi(0) \otimes \tilde{\Pi}(0)$ -module:

$$\mathcal{L}_0(M^\mu(r)) \cong \Pi(0) \otimes \tilde{\Pi}(0).e^{\beta-\delta-(r+1)(\alpha+\beta)+\mu(\delta+\varphi)}.$$

From technical reasons our proof uses assumption that μ is real. But relation (10.48) enables us to present an alternative proof of irreducibility which works for complex parameters μ .

Corollary 10.4. *Assume that $(r, \mu) \in \mathbb{C}^2$ such that $r \notin \mathbb{Z}$, $\mu \notin (\frac{1}{2} + \mathbb{Z})$ and $r - \mu \notin \mathbb{Z}$. Then on the $\Pi(0) \otimes \tilde{\Pi}(0)$ -module*

$$\mathcal{L}_0(M^\mu(r)) = \Pi(0) \otimes \tilde{\Pi}(0).e^{\beta-\delta-(r+1)(\alpha+\beta)+\mu(\delta+\varphi)}$$

exists the structure of an irreducible $\mathbb{Z}_{\geq 0}$ -graded $L_{A_2}(-\frac{3}{2}\Lambda_0)$ -module.

Proof. Let $E_{i,j} := e^{\beta-\delta+(-r-1-i)(\alpha+\beta)+(\mu+j)(\delta+\varphi)}$. Then the lowest component of $\mathcal{L}_0(M^\mu(r))$ is spanned by $\{E_{i,j} \mid i, j \in \mathbb{Z}\}$. By using a direct calculation we get

$$\begin{aligned} e_{\alpha_1}(0)E_{i,j} &= \frac{1}{\sqrt{2}}E_{i,j+1}, \\ e_{\alpha_2}(0)E_{i,j} &= \frac{1}{\sqrt{2}}(1-2\mu-2j)E_{i-1,j-1}, \\ f_{\alpha_1}(0)E_{i,j} &= -\frac{1}{\sqrt{2}}(1-2\mu-2j)(r+i-\mu-j+1)E_{i,j-1}, \\ f_{\alpha_2}(0)E_{i,j} &= \frac{1}{\sqrt{2}}(r+i+1)E_{i+1,j+1}. \end{aligned}$$

So the lowest component of $\mathcal{L}_0(M^\mu(r))$ is an irreducible A_2 -module. Now irreducibility follows easily from relation (10.48) in Zhu's algebra $A(L_{A_2}(-\frac{3}{2}\Lambda_0))$ and similar arguments as in the proof of Theorem 7.1. \square

Remark 10.1. *Modules in Corollary 10.4 don't belong to the category \mathcal{O} . Their lowest components are irreducible sl_3 -modules having all 1-dimensional weight spaces (cf. [BL]).*

Finally we have a family of logarithmic modules:

Corollary 10.5. *For every $\lambda \in \mathbb{C}$ $\widetilde{SV}(\lambda) \otimes F_{-1}$ is an logarithmic $L_{A_2}(-\frac{3}{2}\Lambda_0)$ -module.*

11. CONNECTION WITH $\mathcal{W}_{A_2}(p)$ -ALGEBRAS: $p = 2$

Let $\mathfrak{h} = \mathbb{C}h_{\alpha_1} + \mathbb{C}h_{\alpha_2}$ be the Cartan subalgebra of $\mathfrak{g} = sl_3$. Then the Heisenberg subalgebra of $L_{A_2}(-\frac{3}{2}\Lambda_0)$ generated by \mathfrak{h} is isomorphic to $M_{\mathfrak{h}}(1)$. In this section we shall study the parafermion vertex operator algebra $K(sl_3, -\frac{3}{2}) = \text{Com}(M_{\mathfrak{h}}(1), L_{A_2}(-\frac{3}{2}\Lambda_0))$ and relate it with vertex algebras $\mathcal{W}_{A_2}(p)$ appearing in logarithmic conformal field theory.

We consider the lattice

$$\sqrt{p}A_2 = \mathbb{Z}\gamma_1 + \mathbb{Z}\gamma_2, \quad \langle \gamma_1, \gamma_1 \rangle = \langle \gamma_2, \gamma_2 \rangle = 2p, \quad \langle \gamma_1, \gamma_2 \rangle = -p.$$

Let $M_{\gamma_1, \gamma_2}(1)$ be the standard Heisenberg vertex subalgebra of the lattice vertex algebra $V_{\sqrt{p}A_2}$ generated by the Heisenberg fields $\gamma_1(z)$ and $\gamma_2(z)$.

The vertex algebra $\mathcal{W}_{A_2}(p)$ is defined (cf. [AM4], [S1]) as a subalgebra of the lattice vertex algebra $V_{\sqrt{p}A_2}$ realized as

$$\mathcal{W}_{A_2}(p) = \text{Ker}_{V_{\sqrt{p}A_2}} e_0^{-\gamma_1/p} \bigcap \text{Ker}_{V_{\sqrt{p}A_2}} e_0^{-\gamma_2/p}.$$

We also have its subalgebra:

$$\overline{M_{\gamma_1, \gamma_2}(1)} = \text{Ker}_{M_{\gamma_1, \gamma_2}(1)} e_0^{-\gamma_1/p} \bigcap \text{Ker}_{M_{\gamma_1, \gamma_2}(1)} e_0^{-\gamma_2/p} = W_{A_2}(p) \bigcap M_{\gamma_1, \gamma_2}(1).$$

$\mathcal{W}_{A_2}(p)$ and $\overline{M_{\gamma_1, \gamma_2}(1)}$ have a vertex subalgebra isomorphic to the simple $\mathcal{W}(2, 3)$ -algebra with central charge $c_p = 2 - 24\frac{(p-1)^2}{p}$.

It is expected that $\mathcal{W}_{A_2}(p)$ is a C_2 -cofinite vertex algebra for $p \geq 2$ and that it is a completely reducible $\mathcal{W}(2, 3) \times \mathfrak{sl}_3$ -module. Then as a vertex algebra $\mathcal{W}_{A_2}(p)$ is strongly generated by $\mathcal{W}(2, 3)$ generators and by $\mathfrak{sl}_3 \cdot e^{-\gamma_1 - \gamma_2}$, so by 8 primary fields for the $\mathcal{W}(2, 3)$ -algebra.

Note that $\mathcal{W}_{A_2}(p)$ is a generalization of the triplet vertex algebra $\mathcal{W}(p)$ and $\overline{M_{\gamma_1, \gamma_2}(1)}$ is a generalization of the singlet vertex subalgebra of $\mathcal{W}(p)$.

In the case $p = 2$ we shall embed all vertex algebras defined above in the lattice vertex algebra V_L . Define

$$\gamma_1 = -2\alpha, \quad \gamma_2 = \alpha + \beta - 2\delta.$$

Then $\mathbb{Z}\gamma_1 + \mathbb{Z}\gamma_2 \cong \sqrt{2}A_2$.

Lemma 11.1. *We have:*

$$K(\mathfrak{sl}_3, -\frac{3}{2}) \subset \overline{M_{\gamma_1, \gamma_2}(1)}.$$

Proof. First we notice that:

$$L(-\frac{3}{2}\Lambda_0) \subset \Pi(0) \otimes \tilde{\Pi}(0) \cong \mathbb{C}[\mathbb{Z}(\alpha + \beta) + \mathbb{Z}(\delta + \varphi)] \otimes M_{\mathfrak{h}}(1) \otimes M_{\gamma_1, \gamma_2}(1).$$

This implies that

$$\text{Com}(M_{\mathfrak{h}}(1), L_{A_2}(-\frac{3}{2}\Lambda_0)) \subset M_{\gamma_1, \gamma_2}(1),$$

where $M_{\gamma_1, \gamma_2}(1)$ is identified with the subalgebra $1 \otimes 1 \otimes M_{\gamma_1, \gamma_2}(1)$. The assertion of the lemma follows from fact that e_0^α and \tilde{Q} act trivially on $L(-\frac{3}{2}\Lambda_0)$. \square

By using the following realization of the Weyl vertex algebra

$$M = \text{Ker}_{\Pi(0)} e_0^\alpha$$

(see [A3], [Fr] for details) we conclude that

$$\text{Ker}_{M_{\gamma_1, \gamma_2}(1)} e_0^{-\frac{1}{2}\gamma_1} = M \otimes F \bigcap M_{\gamma_1, \gamma_2}(1).$$

Since $V = \text{Ker}_{M \otimes F} e_0^{-\frac{1}{2}\gamma_2}$, we get

$$\overline{M_{\gamma_1, \gamma_2}(1)} = V \cap M_{\gamma_1, \gamma_2}(1) \subset V^{(0)} \subset L_{A_2}(-\frac{3}{2}\Lambda_0).$$

So we have proved:

Theorem 11.1. *We have:*

$$\overline{M_{\gamma_1, \gamma_2}(1)} \cong K(\mathfrak{sl}_3, -\frac{3}{2}).$$

Remark 11.1. *In the case $g = \mathfrak{sl}_2$ we have $K(\mathfrak{sl}_2, -\frac{1}{2}) \cong W(2, 3)_{-2}$ (cf. [R], [Wn]) and $K(\mathfrak{sl}_2, -\frac{4}{3}) \cong W(2, 5)_{-7}$ (cf. [A4]) where $W(2, 3)_{-2}$ and $W(2, 5)_{-7}$ are singlet vertex algebras realized as subalgebras of triplet vertex algebras $\mathcal{W}(p)$ for $p = 2, 3$. Theorem 11.1 shows that for admissible vertex algebras associated to $\widehat{\mathfrak{sl}}_3$ at level $k = -3/2$ we have interpretation of the coset $K(\mathfrak{sl}_3, k)$ in the framework of vertex algebras which are higher rank generalizations of the triplet vertex algebras.*

12. GENERALIZATIONS AND FUTURE WORK

We shall discuss some possible generalizations of the present work.

Bx Corollary 6.2 we have that the simple $N = 4$ vertex superalgebra with $c = -9$ $L_c^{N=4}$ (denoted here by V) is isomorphic to $\text{Ker}_{M \otimes F} \tilde{Q}$. We have seen that in the case $k+2 = \frac{1}{p}$ affine vertex algebra $L_{A_1}(k\Lambda_0)$ is realized inside of the generalized vertex superalgebra $M \otimes F_{p/2}$ where $F_{p/2} = V_{\mathbb{Z}\frac{p}{2}\delta}$ and we have introduced the following (generalized) vertex algebras:

$$\mathcal{V}^{(p)} = \text{Ker}_{M \otimes F_{p/2}} \tilde{Q}.$$

We conjecture that in this case $\mathcal{V}^{(p)}$ is strongly generated by generators e, f, h of $L_{A_1}(k\Lambda_0)$ and

$$\begin{aligned} \tau_{(p)}^+ &= e^{\frac{p}{2}\delta}, \\ \bar{\tau}_{(p)}^+ &= Qe^{\frac{p}{2}\delta}, \\ \tau_{(p)}^- &= f(0)e^{\frac{p}{2}\delta}, \\ \bar{\tau}_{(p)}^- &= -f(0)Qe^{\frac{p}{2}\delta}. \end{aligned}$$

Remark 12.1. *Drinfeld-Sokolov functor sends $L_{A_1}(k\Lambda_0)$ to the simple Virasoro vertex algebra $L(c_{1,p}, 0)$ with central charge of $(1, p)$ -models. This suggests that $\mathcal{V}^{(p)}$ is mapped to the doublet vertex algebra $\mathcal{A}(p)$ and that a \mathbb{Z}_2 orbifold of $\mathcal{V}^{(p)}$ is naturally mapped to the triplet vertex algebra $\mathcal{W}(p)$. These vertex algebras can be considered as logarithmic extensions of $L_{A_1}(k\Lambda_0)$. It is expected that their representation-categories are connected with Nichols algebras studied by A. M. Semikhatov and I. Yu Tipunin [ST].*

Based on the case $p = 2$ we expect that the following conjecture holds:

Conjecture 12.1. *For every $p \geq 3$ $\mathcal{V}^{(p)}$ has finitely many irreducible modules in the category \mathcal{O} . There exists non-semisimple $\mathcal{V}^{(p)}$ -modules from the category \mathcal{O} .*

In [A4], we studied relations between admissible affine vertex algebra $L_{A_1}(-\frac{4}{3}\Lambda_0)$ and vertex algebras associated to $(1, 3)$ -models for the Virasoro algebra (singlet, doublet and triplet vertex algebras). Some constructions of [A4] were generalized (mostly conjecturally) in [CRW] where the authors found a connection between $(1, p)$ models and Feigin-Semikhatov W -algebras $W_2^{(n)}$.

In our case the realization of the admissible simple affine vertex operator algebra $L_{A_2}(-\frac{3}{2}\Lambda_0)$ also admits a natural generalization. Let $F_{-p/2}$ denotes the generalized lattice vertex algebra associated to the lattice $\mathbb{Z}(\frac{p}{2}\varphi)$ such that

$$\langle \varphi, \varphi \rangle = -\frac{2}{p}.$$

Let $\mathcal{R}^{(p)}$ by the subalgebra of $\mathcal{V}^{(p)} \otimes F_{-p/2}$ generated by $x = x(-1)\mathbf{1} \otimes 1$, $x \in \{e, f, h\}$, $1 \otimes \varphi(-1)\mathbf{1}$ and

$$(12.49) \quad e_{\alpha_1, p} := \frac{1}{\sqrt{2}} \tau_{(p)}^+ \otimes e^{\frac{p}{2}\varphi}$$

$$(12.50) \quad f_{\alpha_1, p} := \frac{1}{\sqrt{2}} \bar{\tau}_{(p)}^- \otimes e^{-\frac{p}{2}\varphi}$$

$$(12.51) \quad e_{\alpha_2, p} := \frac{1}{\sqrt{2}} \bar{\tau}_{(p)}^+ \otimes e^{-\frac{p}{2}\varphi}$$

$$(12.52) \quad f_{\alpha_2, p} := \frac{1}{\sqrt{2}} \tau_{(p)}^- \otimes e^{\frac{p}{2}\varphi}$$

Clearly, $\mathcal{R}^{(2)} \cong L_{A_2}(-\frac{3}{2}\Lambda_0)$. In general, $\mathcal{R}^{(p)}$ is an extension of

$$L_{A_1}((\frac{1}{p} - 2)\Lambda_0) \otimes M_\varphi(1)$$

by 4 fields of conformal weight $p/2$.

We believe that $\mathcal{R}^{(p)}$ for $p \geq 3$ is also part of a series of generically existing vertex (super)algebras.

In order to present some evidence for this statement, we consider the universal affine \mathcal{W} algebras $\mathcal{W}^k(\mathfrak{g}, f_\theta)$ associated with (\mathfrak{g}, f_θ) where \mathfrak{g} is a simple Lie algebra and f_θ is a root vector associated to the lowest root $-\theta$. Let $\mathcal{W}_k(\mathfrak{g}, f_\theta)$ be its simple quotient. Let $\mathfrak{g} = sl_4$. Then by Proposition 4.1. of [KRW], $\mathcal{W}^k(sl_4, f_\theta)$ is generated by 4 four vectors of conformal weight one which generate affine vertex algebra associated to \widehat{gl}_2 at level $k+1$, Virasoro vector and four even vectors of conformal weight $3/2$. By using concepts from [AP] (slightly generalized for \mathcal{W} -algebras) one can easily show that there is a conformal embedding of $L_{gl(2)}(-\frac{5}{3}\Lambda_0)$ into simple vertex algebra $\mathcal{W}_k(sl_4, f_\theta)$. Therefore $\mathcal{W}_k(sl_4, f_\theta)$ for $k = -8/3$ is also an extension of

$$L_{A_1}(-\frac{5}{3}\Lambda_0) \otimes M_\varphi(1)$$

by four fields of conformal weight $3/2$.

This supports the following conjecture:

Conjecture 12.2. *We have:*

$$\mathcal{R}^{(3)} \cong \mathcal{W}_k(sl_4, f_\theta) \quad \text{for } k = -\frac{8}{3}.$$

It is clear that these vertex algebras also admit logarithmic representations and have interesting fusion rules. We plan to investigate the representation theory of these vertex algebras in our forthcoming papers.

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